A Hall–Drift Induced Magnetic Field Instability

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In the presence of a strong magnetic field and under conditions as realized in the crust and the superfluid core of neutron stars the Hall–drift dominates the field evolution. We show by a linear analysis that for a sufficiently strong large–scale background field depending at least quadratically on position in a plane conducting slab an instability occurs which rapidly generates small–scale fields. Their growth rates depend on the choice of the boundary conditions, increase with the background field strength and may reach 10^3 times the ohmic decay rate. The effect of that instability on the rotational and thermal evolution of neutron stars is discussed.

In the presence of a magnetic field the electric conductivity becomes a tensor and, what is more, two non–linear effects are introduced into Ohm's law: the Hall–drift and the ambipolar diffusion. However, if the conducting matter consists of electrons and one sort of ions and no neutral particles take part in the transport processes the ambipolar diffusion is absent [1]. Such a situation is realized e.g. in crystallized crusts of neutron stars and/or in their cores if the neutrons are superfluid, but the protons are normal and the electrons may therefore collide with protons but effectively not with the neutrons.

The effect of the Hall–drift on the magnetic field evolution of isolated neutron stars has been considered by a number of authors (see e.g. [2–8]). They discussed the redistribution of magnetic energy from an initially large–scale (e.g. dipolar) field into small–scale components due to the non–linear Hall–term. Though the Hall–drift is a non–dissipative process, the tendency to redistribute the magnetic energy into small scales may accelerate the field decay considerably.

Indeed, when starting with a large–scale magnetic field the *Hall–cascade* derived in [3] will generate small–scale field components down to a scalelength l_{crit}, where the ohmic dissipation begins to dominate the Hall–drift. Considering numerically the evolution of a magnetic field in a sphere consisting of spherical harmonics up to a multipolarity $l = 5$, Shalybkov & Urpin [6] concluded that the inclusion of higher harmonics will not influence the magnetic evolution. This conclusion is what we want to put in question.

In some of the above–mentioned investigations numerical instabilities are reported if either too many harmonics were taken into account [7] or the initial field is too strong [5]. Also, when considering the thermomagnetic field generation in the crust of young neutron stars [9], where small–scale modes are the first ones to be excited numerical instabilities occurred caused exclusively by the Hall–drift.

Here we want to show that all the observed instabilities are very likely in their essence not of numerical origin but have physical reasons. In presuming that we felt strongly supported by the close analogy of the linearized field evolution equation including Hall–drift to the induction equation including the so–called $\vec{\omega} \times \vec{j}$–effect introduced by Rädler [10]. Within the framework of mean–field dynamo theory (see e.g. [11]) he demonstrated the possible occurrence of magnetic instabilities in an electrically conducting fluid if a shear flow acts together with an electromotive force (e.m.f.) perpendicular to the current density $\vec{j} \propto \operatorname{curl} \vec{B}$.

To prove the existence of a *Hall–drift induced instability* we employ a simplified model: We assume spatial constancy of the conductive properties of the matter and show that under special conditions with respect to the (large–scale) background field strength and geometry an instability occurs which quickly transfers magnetic energy from the background field to small–scale perturbations. We present the result of a linearized analysis which returns only growth rates and the spatial structure of the unstable field modes. Only a fully non–linear analysis is able to yield saturation values of the excited small–scale modes.

The instability may work in different physical systems but is probably most efficient in modifying the field decay in compact astrophysical bodies. Then it will act only during an episode of the field decay, which unavoidably leads to a zero field. This episode, however, may have observable consequences.

In the absence of motions and of ambipolar diffusion, the equations which govern the magnetic field are

$$\dot{\vec{B}} = -c\ \operatorname{curl}\left(\frac{c}{4\pi\sigma}\big(\operatorname{curl}\vec{B} + \omega_B\tau_e\,(\operatorname{curl}\vec{B}\times\vec{e}_B\,)\big)\right) \qquad (1)$$
$$\operatorname{div}\vec{B} = 0\ ,$$

where c is the speed of light, σ the electric conductivity caused by electrons, τ_e the electron relaxation time and $\omega_B = e|\vec{B}|/m_e^* c$ the electron Larmor frequency, with e being the elementary charge and m_e^* the effective mass of an electron. $\vec{e}_B$ is the unit vector in $\vec{B}$–direction. An estimate of the two terms on the r.h.s. of (1) gives rise to the supposition that the Hall–drift becomes important only if $\omega_B\tau_e > 1$.

Using standard arguments one can immediately state that in the absence of currents at infinity the total energy of any solution of (1) is bound to decrease monotonically

to zero since the Hall–term $\propto \operatorname{curl}\vec{B} \times \vec{e}_B$ is unable to deliver energy (nor to consume it).

For simplicity we assume the conductive properties of the matter to be constant in space and time, that is, we assume constant σ and τ_e/m_e^*. Thus, the induction equation can be rewritten in dimensionless variables such that it no longer contains any parameter and the magnetic field evolution is solely determined by its initial configuration $\vec{B}(\vec{x},0)$. For that purpose we normalize the spatial coordinates by a characteristic length L of the model (for a neutron star it could be, e.g., its radius or the thickness of its crust), the time by the ohmic decay time $4\pi\sigma L^2/c^2$ and the magnetic field by $B_N = m_e^* c/e\tau_e$. The governing equations in these dimensionless variables read

$$\dot{\vec{B}} = \Delta\vec{B} - \operatorname{curl}(\operatorname{curl}\vec{B} \times \vec{B}\,)\ ,\ \ \operatorname{div}\vec{B} = 0\ , \qquad (2)$$

where the differential operations have to be performed with respect to the now dimensionless spatial and time coordinates x, y, z and τ, respectively.

Stepping now into the search for instabilities we first have to define a proper reference state $\vec{B}_0$. In order to avoid difficulties in defining the term "instability" and to facilitate the calculations we assume $\vec{B}_0$ to be constant in time. Consequently we are forced to assume the existence of an additional e.m.f. which prevents $\vec{B}_0$ from decaying. Although appearing to be very artificial, we find this measure to be legitimate as long as the results of the stability analysis are applied to real physical situations obeying the constraint that the background field $\vec{B}_0$ is changing only slightly during the considered period of time.

Linearization of (2) about $\vec{B}_0$ yields

$$\dot{\vec{b}} = \Delta\vec{b} - \operatorname{curl}(\operatorname{curl}\vec{B}_0 \times \vec{b}\ + \operatorname{curl}\vec{b} \times \vec{B}_0\,)\ ,\ \ \operatorname{div}\vec{b} = 0 \qquad (3)$$

describing the behavior of small perturbations $\vec{b}$ of the reference state.

With respect to the magnetic energy balance Eq. (3) shows a remarkable difference to Eq. (2). Along with the term $\operatorname{curl}\vec{b} \times \vec{B}_0$ which is again energy–conserving now as a second Hall–term $\operatorname{curl}\vec{B}_0 \times \vec{b}$ occurs which may well deliver or consume energy (from/to $\vec{b}$!) since in general the integral $\int_V (\operatorname{curl}\vec{B}_0 \times \vec{b}) \cdot \operatorname{curl}\vec{b}\, dV$ will not vanish. This reflects the fact that the linearized Hall–induction equation describes the behavior of only a part of the total magnetic field. Actually, perturbations may grow only on expense of the energy stored in the background field. Considering (3), we can determine a scale below which the ohmic dissipation dominates the Hall-drift. Estimating $|\operatorname{curl}\vec{B}_0|$ and $|\operatorname{curl}\vec{b}\,|$ by $\bar{B}_0$ and $\bar{b}/l$, respectively, we find the critical scale of $\vec{b}$ to be $l_{crit} \lesssim 1/\bar{B}_0$, which is de–normalized $l_{crit} \lesssim L/(\omega_{\bar{B}_0}\tau_e)$, identical with the expression derived in [3] considering the Hall–cascade in analogy with the turbulent flow of an incompressible fluid.

Let us now specify the geometry of our model and the background field. We consider a slab which is infinitely extended both into the x– and y–directions but has a finite thickness $2L$ in z–direction. The background field is assumed to be parallel to the surface of the slab pointing, say, in x–direction and to depend on the z coordinate only, i.e. $\vec{B}_0 = f(z)\vec{e}_x$. Guided by the conditions under which the above–mentioned magnetic instability [10] may work, we conclude that $f(z)$ has to be at least quadratic thereby ensuring that the first term in (3) is able to play the role of the shear flow, the second the role of the $\vec{\omega} \times \vec{j}$–term. Note, that by this choice $\operatorname{curl}\vec{B}_0 \times \vec{B}_0$ represents a gradient. Thus the unperturbed evolution of the background field is not at all affected by the Hall–drift; in the absence of an e.m.f. it would decay purely ohmically!

Further on we decompose a perturbation $\vec{b}$ into a poloidal and a toroidal component, $\vec{b} = \vec{b}_p + \vec{b}_t$, which can be represented by scalar functions S and T, respectively, by virtue of the definitions

$$\vec{b}_p = -\operatorname{curl}(\,\vec{e}_z \times \nabla S)\ ,\ \vec{b}_t = -\vec{e}_z \times \nabla T\ , \qquad (4)$$

ensuring $\operatorname{div}\vec{b} = 0$ for arbitrary S, T.

For the sake of simplicity we will confine ourselves to the study of plane wave solutions with respect to the x– and y–directions, thus making the ansatz

$$\left\{ \begin{matrix} S \\ T \end{matrix} \right\} (\vec{x},\tau) = \left\{ \begin{matrix} s \\ t \end{matrix} \right\} (z) \exp\left(i\tilde{\vec{k}}\tilde{\vec{x}} + p\tau\right)\ , \qquad (5)$$

where $\tilde{\vec{k}} = (k_x, k_y)$, $\tilde{\vec{x}} = (x, y)$ and p is a complex time increment. It guarantees as well the uniqueness of the poloidal–toroidal decomposition since from $\underset{\sim}{\Delta}(S, T\,) = 0$ it follows $(S, T\,) = 0$ with $\underset{\sim}{\Delta}$ being the 2–dimensional lateral Laplacian (see [12]). With (5) we obtain from (3) two coupled ordinary differential equations

$$\begin{aligned} pt - t''\ + \tilde{k}^2 t \qquad\qquad &= \quad ik_x f(s'' - \tilde{k}^2 s) - ik_x f'' s \\ ps - s'' + \tilde{k}^2 s - ik_y f' s &= -ik_x f t\ , \end{aligned} \qquad (6)$$

where the dash denotes the derivative with respect to z. Together with appropriate boundary conditions Eqs. (6) define an eigenvalue problem with respect to p.

We consider two types of boundary conditions (BC): For the vacuum condition we assume $\operatorname{curl}\vec{B} = \vec{0}$ outside the slab and require continuity of all components of $\vec{B}$ across the boundary. For the perfect–conductor condition an electric field must be prevented from penetrating into the region outside the slab, that is, the normal magnetic and tangential electric field components must vanish at the boundary. In terms of the scalars s and t this means $[s] = [s'] = t = 0$ for the vacuum condition and $s = t' = 0$ for the perfect conductor condition where [.] denotes the jump across a boundary. For $t' = 0$ to be valid the vanishing of $\vec{B}_0$ at the boundary is required.

Making use of the vacuum solutions vanishing at infinity for either halfspace, $z \geq 1$ and $z \leq -1$, respectively, the vacuum boundary condition for s can be expressed as $s' = \mp\tilde{k}s$ for $z = \pm 1$, with $\tilde{k} = |\vec{\tilde{k}}|$.

Obviously, three distinguishable combinations of the boundary conditions are possible: vacuum on either side (VV), perfect conductor on either side (PP), vacuum on one and perfect conductor on the other side of the slab (PV). The latter choice comes closest to neutron star conditions if we think of the crust being neighboured upon a superconducting core on the one and a region with very low conductivity on the other side. The VV boundary condition may in turn be appropriate for a galactic disc. Since both the PV and the VV BC were to be considered, we choose sufficiently curved background field profiles, which obey them, i.e. $f(z) = B_0(1+z)(1-z^2)$ and $f(z) = B_0(1-z^2)$ for BC=PV and BC=VV , respectively.

For certain ranges of the wave numbers k_x, k_y and for $B_0 \gtrsim 3$ we found eigenvalues p with a positive real part, i.e. exponentially growing perturbations. The dependence of the growth rate $\Re(p)$ on the wave numbers for $B_0 = 1000$ and BC = PV is shown in Fig. 1.

Figure 2 shows the dependence of growth rate and wave number k_x of the fastest growing mode on B_0.

An interesting feature is, that the maximum growth rates occur for all B_0 considered at $k_y = 0$. Of course this asymmetry is due to the choice of the background field: once it was chosen parallel to the y–direction the maximum growth rates would occur at $k_x = 0$. Another interesting result is the dependence of the growth rate on the boundary conditions. BC = PV yields the largest values, by a factor $1.6 \ldots 3$ larger than for BC = VV, while BC = PP results in very small growth rates. Note that the most unstable eigenmodes are always non–oscillatory, though oscillating unstable ones exist.

Evidently, the obtained growth rates are in agreement with the constraint, formulated above: In comparison with the background field decay the growth of the most unstable perturbations is a fast process; thus we may consider it as 'episodically unstable'.

With respect to the asymptotic behavior $\sigma \to \infty$ for a fixed (unnormalized!) background field one has to note that the time increment p is normalized on the ohmic decay rate ($\propto \sigma^{-1}$). From Figure 2 it can be inferred $\Re(p) \propto B_0^q$, $q < 1$ for $B_0 \geq 100$, which means that in the limit of negligible dissipation the growth rate in physical units tends to zero.

Figure 3 shows the eigensolutions $(s, t)(z)$ of the fastest growing mode for three different values of B_0 and BC = PV. One can observe that with increasing B_0 the toroidal field becomes more and more small–scaled and concentrated towards the vacuum boundary. In contrast, the corresponding poloidal field remains large–scaled.

The magnetic field structure of the fastest growing mode for $B_0 = 2000$ and BC = PV is shown in Fig. 4.

Clearly, any assignment of the results gained by help of a very simplified model to astrophysical objects has to be done with great care. Even when accepting the plane layer as a reasonable approximation of a neutron star's crust one has to concede that the very specific profiles of $\vec{B}_0$ assumed above may only exemplify the field structure in the crust.

An acceptable approximation of the radial profile of a dipolar crustal field as given e.g. in [13] will in general have to allow for a linear part and non–zero values at the boundaries. Moreover, the strong dependence of the conductive properties on the radial co–ordiante should anyway be taken into account.

To get an impression of possible consequences for the evolution of neutron stars we now simply assume, that the real $\vec{B}$–profile is sufficiently "curved" (i.e. its second derivative is big enough) and associate the parameter B_0 with a typical value of the field.

Assuming further electric conductivity and chemical composition to be constant, $\sigma = 5 \times 10^{26}\mathrm{s}^{-1}$ and the relative atomic weight $A/Z = 25$, respectively, we find the normalization field at a density $\rho = 10^{14}\mathrm{g\ cm}^{-3}$ to be 7×10^{10}G (see e.g. [13]). That is, for typical (inner) crustal magnetic fields ranging between 7×10^{12}G and 1.4×10^{14}G we find a B_0 between 100 and 2000 and the e–folding time of the most rapidly growing unstable mode to be 0.0035 and 0.0003 times the Ohmic decay time, respectively. Thus, an initial perturbation will quickly evolve to a level at which the linear analysis is no longer feasible, that is, at which it starts to drain a remarkable amount of energy out of the background field.

We want to emphasize again that a sufficient curvature of the background field profile is a necessary condition for the occurrence of an unstable behavior. Therefore neither the derivation of the well–known helicoidal waves (whistlers) nor its modification presented in [8] could reveal it because a homogeneous background field is used.

With even more care we may speculate about possible observational consequences. The instability discussed here may perhaps act effectively in the crust of not too young (age $t \gtrsim 10^5$ yrs) neutron stars. For those stars, the small–scale field modes initially generated or existing in the crust have already been decayed and the magnetic field is concentrated almost completely in the large–scale, say, dipolar mode. Simultaneously, in the process of cooling the coefficient $m_e^* c/e\tau_e$ becomes smaller and smaller until the nonlinear Hall term in (2) dominates the linear ohmic term. From that moment the Hall–instability may rise small–scale modes down to scale lengths $\gtrsim l_{crit}$ on expense of the dipolar mode. This would lead to a change of the spin–down behaviour of isolated neutron stars. Deviations from the "standard" rotational evolution will occur when the dipolar field decreases rapidly due to the instability. This may lead to the observation of braking indices $n > 3$ [14] during the action of that instability. Another possible observational consequence is due to enhanced Joule heating, which will keep the neutron star warmer than standard cooling calculations predict after an age critical for the onset of the Hall–instability

($\gtrsim 10^5$ yrs). Third, the strong small-scale field components cause strong small–scale Lorentz forces which may be able to crack the crust. This could be observable in glitches or, depending on the available energy even in Gamma– and X–ray bursts [15].

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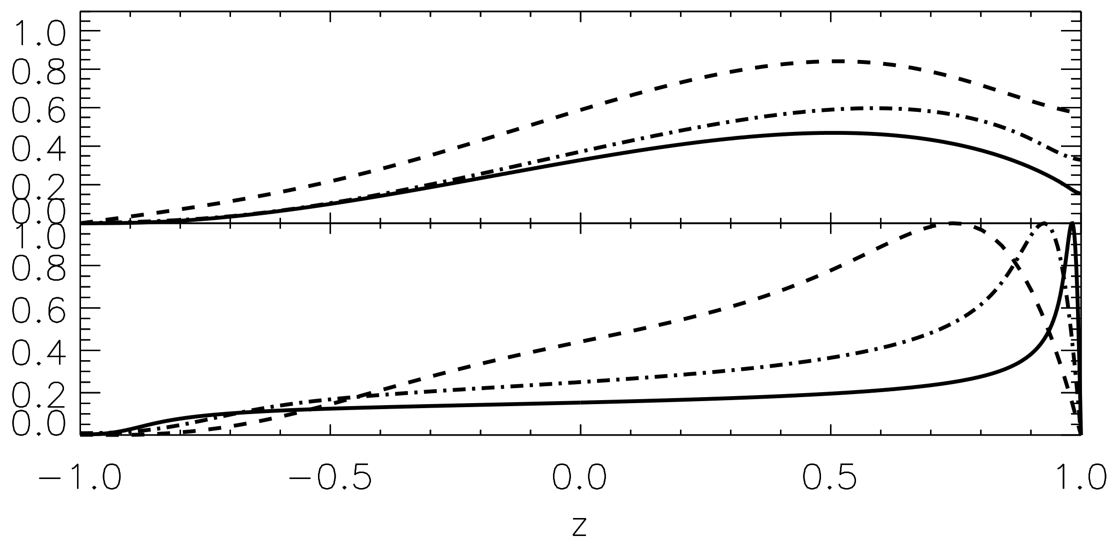
FIG. 1. Growth rate as a function of k_x and k_y for $B_0 = 1000$ and BC = PV. Negative values were set to zero.

FIG. 2. Growth rate and wave number $k_x^{\max}$ of the fastest growing mode as functions of B_0. Solid and dashed lines correspond to BC = PV and BC = VV, while thick and thin lines correspond to growth rates and k_x, respectively.

FIG. 3. Moduli of $(s,t)(z)$ of the fastest growing mode; BC = PV. Solid, dash–dotted and dashed lines refer to $B_0 = 2000$, $B_0 = 100$ and $B_0 = 10$, respectively.

FIG. 4. Field structure of the fastest growing mode for $B_0 = 2000$ and BC = PV. Arrows: $b_{x,z}$, grey shading: value of b_y, dark — into, light — out of the plane.



|s|
|t|
z
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
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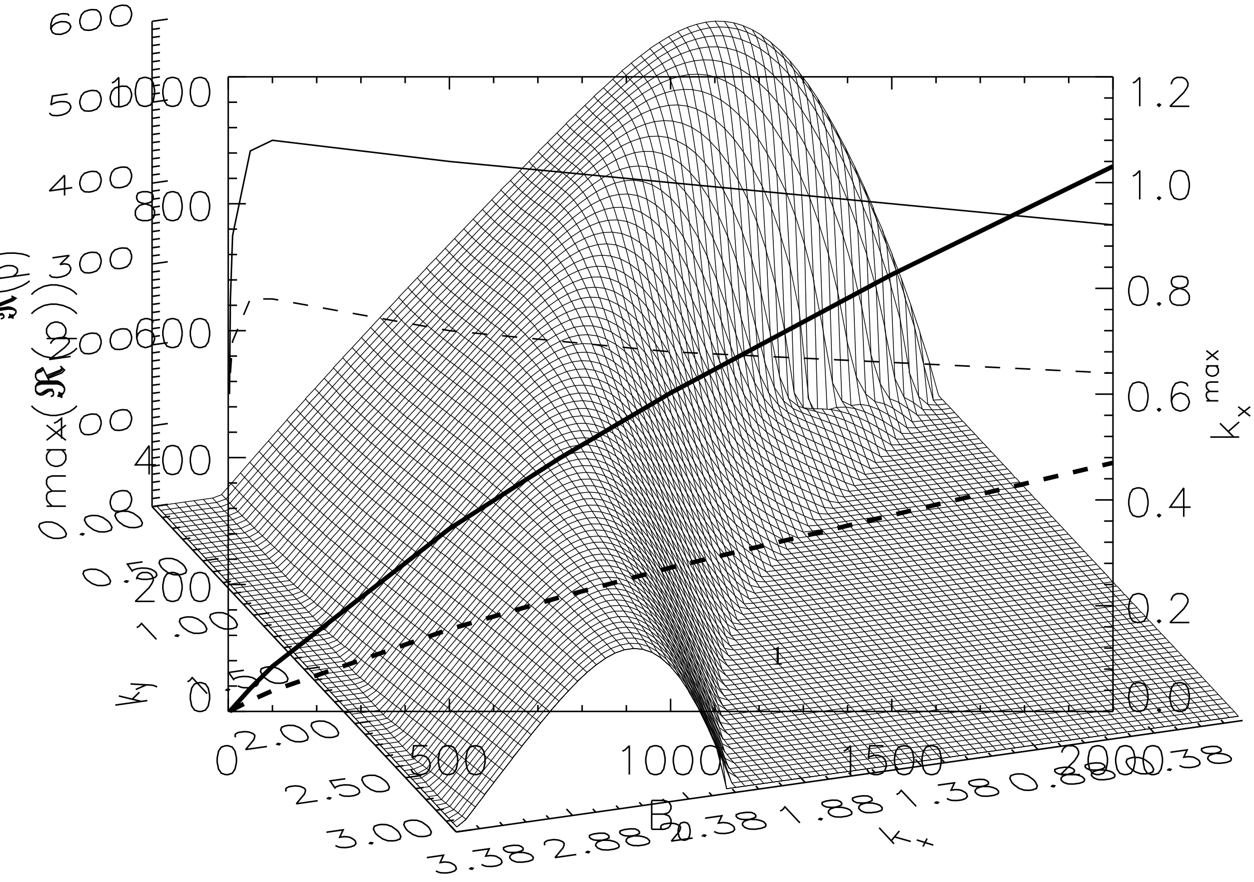
0.5
1.0



ℜ(p)
max(ℜ(p))
k_x^max
k_y
k_x
B_z
600
500
400
300
200
1000
800
600
400
200
0
500
1000
1500
2000
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.00
0.50
1.00
1.50
2.00
2.50
3.00
3.38
2.88
2.38
1.88
1.38
0.88
0.38